

Nonequilibrium thermodynamics and Fisher information: Sound wave propagation in a dilute gasS. P. Flego,¹ B. R. Frieden,² A. Plastino,^{1,2} A. R. Plastino,^{1,3,4} and B. H. Soffer^{2,5}¹Universidad Nacional de La Plata, C.C. 727, 1900 La Plata, Argentina²Optical Sciences Center, University of Arizona, Tucson, Arizona 85721, USA³Argentine National Research Center (CONICET), C.C. 727, 1900 La Plata, Argentina⁴Department of Physics, University of Pretoria, Pretoria 0002, South Africa⁵University of California at Los Angeles, 665 Bienvenida Avenue, Pacific Palisades, California 90272, USA

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As recently shown, a constrained Fisher-information extremizing (CFIE) process is able to deal with both equilibrium and nonequilibrium thermodynamic processes, thus being able to reproduce results deduced by a recourse to Boltzmann's transport equation (BTE). Here, we discuss the propagation of sound waves in a dilute gas and compare the ensuing CFIE solutions with those obtained by a recourse to Grad's approach to the BTE. The final molecular distribution function arrived at is the same following two alternative routes, either (i) the BTE via the Grad approach or (ii) the constrained Fisher treatment that does *not* require the use of the BTE. The way the necessary *a priori* information is used in these two instances, is however, quite different.

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I. INTRODUCTION

Information theory, as developed by Jaynes [1], maximizes Shannon's information entropy [2] subject to constraints which constitute the "prior information." This methodology yields an optimal distribution consistent with these constraints, although the procedure does not always lead to an adequate distribution function [3]. This fact has encouraged the formulation of alternative variational procedures [3–5] that maintain Jaynes' spirit but use, instead, the information measure of Fisher [6]. The Fisher statistics provides an information measure I which, like that of Shannon's, is a functional of the distribution f .

It is becoming increasingly evident [3–5,7–24] that Fisher's information measure I provides a new viewpoint within the information theoretic approach to physics often referred to as "the Wheeler program" [25]. In Refs. [10,14,26] one appreciates the fact that the I concept lays the foundations of thermodynamics *in the usual equilibrium* case. *Nonequilibrium* thermodynamics is amenable to a similar treatment [10,14,27].

The main result of Ref. [27] is the establishing, via I , of a "connection" between nonequilibrium thermodynamics and a Schrödinger-like equation. It is thus seen that both thermodynamics and quantum mechanics can be expressed by a recourse to a formal Schrödinger wave equation (SWE), out of a common informational basis [14]. The physical meaning of this Schrödinger wave equation is quite flexible. In the Fisher instance, its "potential function" $U(x)$ originates in prior data $\langle A_k \rangle_t$, to be chosen according to the application under consideration [14,27]. The $\langle A_k \rangle_t$ are introduced into the theory as *empirical* inputs [27]. Both nonequilibrium and equilibrium thermodynamics can be obtained from this constrained Fisher extremization process whose output is a Schrödinger-like wave equation. Equilibrium thermodynamics corresponds to the ground state (g.s.) solution, while nonequilibrium thermodynamics corresponds to linear superpositions of this g.s. with excited state solutions [27]. In the present paper we shall study, within such a Fisher context,

the propagation of sound waves in a monatomic gas. Our main objective is to compare the ensuing solutions with those obtained using Boltzmann's transport equation.

It is worth mentioning here that some interesting books have recently discussed the connection between information theory, kinetic theory, and nonequilibrium thermodynamics, which is the topic of our present initial endeavors, in which we explore links with Fisher (rather than with Shannon) information. We refer the interested reader to Refs. [28–30]. Some older but quite valuable works are presented in Refs. [31–35]. These books, in addition, summarize the many scores of papers that, over time, laid the foundations for and built the connections with Shannon's information.

The work is organized in the following way. In Sec. II, we review the Fisher information approach to nonequilibrium states via a Schrödinger wave equation [26,27]. For didactic reasons, we will also discuss here the conventional Boltzmann treatment of the sound-propagation problem as a way of getting a proper perspective for our approach. This is done in two steps: (i) in Sec. III, we deal with the Boltzmann transport equation and its conservation theorems (see the Appendix), from which we can construct the initial conditions for our specific problem and (ii) we devote Sec. IV to a spatial method of tackling the Boltzmann transport equation via Gauss-Hermite polynomials by recourse to the Grad approach [36]. The methodology of this section constitutes the stepping stone towards the Fisher information treatment. After this Boltzmann detour, we review in Sec. V the connection between excited eigensolutions to the Schrödinger wave equation and nonequilibrium thermodynamic states [27], and later tackle sound propagation with this technique. Finally, some conclusions are drawn in Sec. VI.

II. DESCRIPTION OF NONEQUILIBRIUM STATES VIA A SCHRÖDINGER WAVE EQUATION

In this section, we review the formalism developed in Ref. [27], regarding the connection between Fisher information and thermodynamics. The latter can be regarded, from

an axiomatic viewpoint, as a logical mathematical structure whose axioms are empirical results [37]. This gives thermodynamics a unique epistemological status among scientific theories. Of course, thermodynamics does not assume any underlying microscopic picture (it does not need it).

Consider now a system that is specified by a set of M physical parameters Θ_k measured at time t . We can write

$$\Theta_k = \langle A_k \rangle_t \quad \text{with} \quad A_k = A_k(x) \quad (k=1, \dots, M) \quad (1)$$

and Θ_k measured at the time t . Note that the set of Θ_k values is the prior knowledge which represents empirical information measured at fixed time t .

Let the pertinent probability distribution function (PDF) to be $p(x|t)$, then by definition we have

$$\langle A_k \rangle_t = \int dx A_k(x) p(x|t), \quad k=1, \dots, M. \quad (2)$$

These mean values play the role of thermodynamical variables, as explained in Ref. [27]. It was shown in Ref. [27] that the relevant PDF $p(x|t)$ in this context extremizes the so-called Fisher information measure I subject to (i) the prior conditions (1) and, of course (ii) the normalization condition

$$\int dx p(x|t) = 1. \quad (3)$$

In order to find this PDF, one has to solve a SWE [5,27,38]

$$-\frac{1}{2} \psi'' - \frac{1}{8} \sum_{k=1}^M \lambda_k(t) A_k \psi = \frac{1}{8} \alpha \psi, \quad (4)$$

where the Lagrange multiplier $\alpha/8$ plays the role of an energy eigenvalue E_n and the sum of the $(\lambda_k A_k)$ is an effective potential function

$$U = U(x, t) = \frac{1}{8} \sum_{k=1}^M \lambda_k(t) A_k(x), \quad (5)$$

where the Lagrange parameters λ_k are fixed by recourse to the available prior information. The modulus squared of the solutions ψ yields the PDF [5,27]

$$|\psi(x, t)|^2 = p(x|t). \quad (6)$$

It is important to note the following points.

(1) No specific potential has been assumed, as is appropriate for thermodynamics. Also, we note that U is a time-dependent potential function and will allow for the description of nonequilibrium situations.

(2) The specific $A_k(x)$ to be used depend upon the nature of the physical application at hand. This application could be of either a classical or a quantum nature.

(3) Equation (4) represents a boundary value problem, generally with multiple solutions, in contrast with the unique solution one obtains when employing the Jaynes-Shannon entropy in place of Fisher's measure [39].

(4) The solution leading to the lowest I value is the equilibrium one [26], while linear superpositions of excited solutions yield the nonequilibrium states [27].

The connection between the Schrödinger wave equation excited solutions and the nonequilibrium thermodynamics will be reviewed in Sec. V.

III. BOLTZMANN TRANSPORT EQUATION FOR SOUND WAVES SPREADING IN A MONATOMIC GAS

The dynamical equation of motion for gases, i.e., the Boltzmann transport equation (BTE), is one of the most useful tools for dealing with nonequilibrium phenomena. This equation makes a clear separation between the effects of the irregular collisions and the effects of regular motion. Regular motion is slowly varying, while collisions are rapid. The Boltzmann transport equation averages out the fast, irregular collisions. Therefore, the scales of time and length of interest in this equation are much larger than those for the collision times and molecular sizes, respectively.

The Boltzmann transport equation for sound waves propagating in a monatomic gas takes the specific form [40,41]

$$\left[\frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{r}} \right] f = D_c f, \quad (7)$$

where $f = f(\mathbf{r}, \mathbf{v}, t)$ is the molecular distribution function (MDF), $D_c f$ represents changes in f due to collision effects (the explicit expression can be found in Ref. [40]), and no external forces \mathbf{F} are assumed to be operative.

In order to investigate nonequilibrium phenomena it is necessary to solve the Boltzmann transport equation with given initial conditions so as to obtain the distribution function as a function of time. Some important properties of any solution to the Boltzmann equation may be obtained from the fact that in any molecular collision there are dynamical quantities that are rigorously conserved. This gives rise to important conservation theorems (see the Appendix). Let $\xi(\mathbf{r}, \mathbf{v})$ be any quantity associated with a molecule of velocity \mathbf{v} located at \mathbf{r} . If, in any collision $\mathbf{v}_1, \mathbf{v}_2 \rightarrow \mathbf{v}'_1, \mathbf{v}'_2$ taking place at \mathbf{r} , we have $\xi_1 + \xi_2 = \xi'_1 + \xi'_2$, we call ξ a conserved quantity and [41]

$$\int d^3v \xi(\mathbf{r}, \mathbf{v}) [D_c f] = 0. \quad (8)$$

From Eq. (8) and an appropriate choice of ξ (see the Appendix), we get three independent conservation theorems for mass, momentum, and energy, respectively [41]. These conservation theorems will be needed below.

In this section, we are concerned with a particular approach to the Boltzmann equation, which was introduced in Ref. [36]. In this respect, it may be of utility to present first a brief recapitulation of the pertinent conservation theorems, illustrated with reference to sound propagation.

A. The concept of local thermal equilibrium

We introduce now the concept of local thermal equilibrium [42]. This is a quite useful concept for a variety of

purposes [40]. It is enough to mention here the description of radiative processes in stellar atmospheres [42]. Consider then a gas of N molecules of mass m in a volume V that, although not in equilibrium, is not far from it. More specifically, and invoking the idea of local thermal equilibrium (LTE), we assume that in the vicinity of any point in the gas, the molecular distribution function is locally the Maxwell-Boltzmann distribution $f^{(0)}(\mathbf{r}, \mathbf{v}, t)$ and that (i) the density $\rho = mn$ [with $n(\mathbf{r}, t) = \int d^3v f(\mathbf{r}, \mathbf{v}, t)$], (ii) the temperature T , and (iii) the average (or bulk) velocity $\mathbf{u} \equiv \langle \mathbf{v} \rangle$ vary only slowly in both space and time. The effect of collisions is not neglected in such an approach. On the contrary, it is regarded as very important, since one assumes that collisions quickly restore local equilibrium. However, the effect of collisions is here of a tacit character, not an explicit one (no appeal to scattering cross sections is made). In these conditions, the idea of local thermal equilibrium entails setting

$$f(\mathbf{r}, \mathbf{v}, t) \simeq f^{(0)}(\mathbf{r}, \mathbf{v}, t), \quad (9)$$

$$f^{(0)}(\mathbf{r}, \mathbf{v}, t) = n(\mathbf{r}, t) (a/\pi)^{3/2} \exp(-a[\mathbf{v} - \mathbf{u}]^2),$$

$$a = (m/2)kT, \quad \text{with } k_B \text{ the Boltzmann constant.} \quad (10)$$

Within the LTE context, the BTE conservation theorems now become equations restricting the behavior of ρ (or of n), T , and \mathbf{u} . Indeed, one easily ascertains [41] that the quantity $P_{ij} \equiv \rho \langle [(\mathbf{v})_i - (\mathbf{u})_i][(\mathbf{v})_j - (\mathbf{u})_j] \rangle$, i.e., the pressure tensor, becomes the hydrostatic pressure $P = nk_B T$. Additionally, we find that the conservation theorems take the following form [41]:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \quad (\text{continuity}) \quad (11)$$

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \mathbf{u} = - \frac{1}{\rho} \nabla P \quad (\text{Euler}) \quad (12)$$

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) T = - \frac{1}{c_V} (\nabla \cdot \mathbf{u}) T, \quad (13)$$

with c_V the specific heat at constant volume ($c_V = 3/2$). These equations must be obeyed by the slow, smooth variation in space and time, which according to the LTE, the quantities ρ , $n(\mathbf{r}, t)$, T , and \mathbf{u} undergo.

It is important, for the Fisher considerations to be undertaken below, to point out that Eqs. (11)–(13) can be also derived from heuristic considerations of a more general character, without a recourse to the LTE [41]. Additionally, and this is the central point that underlies our Fisher approach, they can be confirmed as *experimental results* from the field of hydrodynamics [41]. Indeed, Eqs. (11)–(13) are the hydrodynamic equations for the nonviscous flow of a fluid and possess solutions describing flow patterns that persist indefinitely [41]. In the local thermal equilibrium framework, the local Maxwell-Boltzmann distribution never decays to the true (global) Maxwell-Boltzmann distribution, which is in rough agreement with experience.

The quantity $(\partial/\partial t + \mathbf{u} \cdot \nabla)X$ is called the “material (or convective) derivative of X ” [41]. It is the time rate of change of X to an observer moving with the velocity \mathbf{u} . This observer is said to be moving along a streamline. One important consequence of the above equations (see the Appendix) is that *diluted monatomic gases undergo only adiabatic transformations* when observed from a reference frame moving along the streamline [41]. This entails that the equation of state reads

$$P\rho^{-5/3} = \text{const (along a streamline)}. \quad (14)$$

Summing up, the results recapitulated in this section provide important information that is, however, contemplated in a different way according to whether one is to treat the sound-propagation problem following the prescriptions of (A) the Boltzmann transport equation (via the method of moments technique [40,36,43], see below) or (B) a constrained Fisher extremizing process.

Accordingly, it is important to realize that everything we learn from Eqs. (11)–(13), together with Eq. (14), is regarded by the two methodologies discussed in this paper in quite *different* ways, namely, either as in (A) Grad’s method, where the f_0 form is introduced into the BTE conservation theorems, or (B) in the CFIE approach which deals with *experimental* information pertaining to the field of hydrodynamics.

Obviously, the second approach, being an information theory technique, needs empirical informational input that the Grad method can dispense with, replacing it by theoretical considerations. The two different paths will be shown below to lead to the same molecular distribution, though. One ends up entering information into each of the methods’ equations (that are quite different indeed) using the same mathematical input. The epistemological origin of this input, let us insist on this point once again, is different in the two cases.

B. Propagation of sound in a dilute gas

Sound can be thought of as a perturbation of the medium that manifests itself via small departures, from their global-equilibrium values, of various fields: those of (i) density $\rho(\mathbf{r}, t) = mn(\mathbf{r}, t)$, (ii) pressure $P(\mathbf{r}, t)$, (iii) temperature $T(\mathbf{r}, t)$, and (iv) fluid bulk velocity $\mathbf{u}(\mathbf{r}, t) = [\int d^3v \mathbf{v} f(\mathbf{r}, \mathbf{v}, t)/n(\mathbf{r}, t)]$. The corresponding global-equilibrium values are denoted by $\rho_o = mN/V = mn_o$, P_o , T_o , and \mathbf{u}_o , respectively. In the present instance, $\mathbf{u}_o = 0$, since the gas is globally at rest. Notice that the superscript “(0)” is used for the LTE. The subscript “o” indicates that we deal with global-equilibrium quantities.

Sound propagation in a dilute gas entails planar motion of the medium along, say, the z direction, so that $\mathbf{u} = u\check{e}_z$. The pertinent analysis starts by representing both the gas density ρ and its pressure P in the form of sums of global-equilibrium values plus perturbations that are produced by the sound wave, i.e.,

$$\rho \equiv \rho^{(0)} = \rho_o(1 + \sigma); \quad P \equiv P^{(0)} = P_o(1 + \mathcal{P}). \quad (15)$$

Due to the fact that the perturbation is small, ρ and P will slightly differ from their respective (global) equilibrium values. Also, as the bulk velocities \mathbf{u} are much smaller than the sound velocity c_o , we have

$$\sigma \sim \mathcal{P} \sim \frac{|u|}{c_o} \sim \mu \ll 1, \quad (16)$$

μ being a small dimensionless parameter. In these conditions, Eq. (14) entails

$$T = \left(\frac{\rho}{\rho_o} \right)^{2/3} T_o. \quad (17)$$

Notice that quite an important amount of information is encapsulated into this equation of state. On one hand, this equation arises from the conservation theorems. On the other one, it can also be regarded as an experimental result pertaining to the study of nonviscous fluids [41]. This last point of view will be invoked in our Fisher treatment below.

Writing now ρ as a function of σ [cf. Eq. (15)], explicit introduction of the temperature T , given by Eq. (17), into the conservation expressions (11)–(13) leads [in agreement with Eq. (15)] to a set of just two equations, namely,

$$\sigma_t + u\sigma_z + (1 + \sigma)u_z = 0, \quad (18)$$

$$u_t + uu_z + c_o^2(1 + \sigma)^{-1/3}\sigma_z = 0, \quad (19)$$

where $\sigma_z = \partial\sigma/\partial z$, $\sigma_t = \partial\sigma/\partial t$ (similar notation for u derivatives), and c_o is given by

$$c_o = \sqrt{\frac{5}{3} \frac{k_B T_o}{m}}. \quad (20)$$

An appropriate treatment of the set of Eqs. (18) and (19) [41] allows one to obtain linear equations for σ and u when second-order quantities in μ are neglected. With such a first-order approximation (in μ), a straightforward manipulation of the above equations leads to the following results [41].

(i) For the σ field, we have

$$\sigma_{zz} - \frac{1}{c_o^2} \sigma_{tt} = 0, \quad (21)$$

which is a wave equation describing a sound wave of amplitude σ_o with propagation velocity c_o ,

$$\sigma = \sigma_o \exp[i(kz - w_o t)], \quad c_o = \frac{w_o}{k}. \quad (22)$$

(ii) For the u field one finds an equation of form (21) so that, keeping in mind Eq. (16), we can write $u = c_o \sigma$. In this way, up to the first order in μ we have

$$\rho = \rho_o(1 + \sigma); \quad u = c_o \sigma; \quad T = T_o(1 + \frac{2}{3}\sigma), \quad (23)$$

with σ given by Eq. (22).

We have thus obtained, on the basis of (i) the conservation theorems within a LTE context, and (ii) a first-order approximation in μ , an explicit expression for $\sigma(z, t)$ and therefore for

$$\rho(z, t) = \rho_o[1 + \sigma(z, t)]. \quad (24)$$

What is then the form of the corresponding molecular distribution function? It should be of the form

$$f(z, v, t) = f_o(v)[1 + g(v)\sigma(z, t)], \quad (25)$$

with $g(v)$ an appropriate function of the molecular velocities v . To find it we need to solve the BTE. The path we shall follow in order to answer the above question is the first significant result of the present paper.

At this, stage it is important to note that if Eq. (22) for σ is inserted into the MDF (evaluated up to some appropriate perturbation order), the resulting expression does not *explicitly* include collision effects. In this effort we will be concerned with this type of MDF. A better approximation for σ , consistent with Eq. (23) but explicitly incorporating such effects, would be clearly desirable. Steps towards such a goal are under consideration by the present authors and will be published in the near future. We pass now to a methodology for dealing with the transport equation.

C. The method of moments technique

In dealing with the Boltzmann transport equation one often faces great difficulties whose character is of both mathematical and physical natures. The D_{cf} contribution depends on the effective cross section for molecular collision, σ' . As one often ignores the form of σ' , the problem of integrating the Boltzmann transport equation becomes problematic indeed. In practice, one makes use of simplified models for which the molecular interaction can be represented by a sufficiently simple law that depends on the distance among the particles. A number of methods to tackle the ensuing, simplified Boltzmann problem have been developed over the years. In this respect, an interesting approach to nonequilibrium thermodynamics via the Boltzmann transport equation has been developed by Grad [36] (see also Refs. [40,43]). Due to the fact that the concomitant treatment allows one to visualize the connection between the Schrödinger wave equation's excited states and the Boltzmann transport equation's solutions, we will use below the Grad technique to find the distribution function for the sound-propagation problem.

IV. METHOD OF MOMENTS TECHNIQUE TO NONEQUILIBRIUM THERMODYNAMICS

A. Brief review

Consider the nonequilibrium state of a gas after the time t has elapsed, which is large compared to the time of initial randomization (this time t is regarded as *fixed*) such that conditions of local thermal equilibrium [42] prevail, a circumstance very common indeed [40,42]. Now, at each point of the vessel containing the gas a state arises which is close to the *local* equilibrium state. This is, of course, the local

thermal equilibrium scenario that allows us to use its associated conservation laws. More specifically, we consider the sound-propagation stage described in Sec. III B [cf. Eq. (15)], and assume a global f_o , the Maxwell-Boltzmann law on velocities, as envisioned there. This allows one to expand the nonequilibrium distribution f as

$$f=f(x,t)=f_o(x)[1+\epsilon\phi(x,t)], \quad (26)$$

where ϵ is small. Determining the function ϕ will be our goal now. Following elementary ideas from the theory of vector spaces, the unknown function $\phi(x,t)$ may itself be expanded in an appropriate basis. The (orthogonal) Hermite-Gaussian polynomials $H_i(x)$ that we will employ here usually constitute quite a convenient basis in many instances, especially in quantum mechanical calculations. Defining $a_i(t)$ as the pertinent coefficients at the fixed time t , we have

$$\phi(x,t)=\sum_i a_i(t)H_i(x). \quad (27)$$

Expansion (27) was proposed by Grad in 1949, and it is well known in the kinetic theory [36]. It is important to point out that Hermite-Gaussian polynomials are orthogonal with respect to a Gaussian kernel, i.e., the *equilibrium distribution*. No other set of functions is orthogonal (and complete) with respect to a Gaussian kernel function.

Now, because of orthogonality, the unknown coefficients $a_i(t)$ relate linearly to appropriate (unknown) moments of f over velocity space (x space), so that substituting the expansion for f into the transport equation and integrating over all velocities yields now a set of equations in the moments (which are generally a function of the fixed time value t). These equations are solvable subject to *known initial conditions*, so that these moments now become known (*including any time dependence*). As a consequence, the coefficients $a_i(t)$ of Eq. (27) are also known, which gives f .

According to Refs. [40,36,43], the solution of the above system of equations would be equivalent to the exact solution of Boltzmann's equation (if enough *a priori* information were available).

B. The treatment of sound propagation by the method of moments

We tackle now our BTE treatment of sound propagation. We will do this using the Grad-MOM algorithm. Remember that, to such an end, we need the results obtained in Sec. III. We start by casting $f_o=\prod_i^3 f_{o,i}$ and focus our attention on the z component of f , the direction of propagation. The Grad-method of moments (MOM) to find f [40,36,43] entails its expansion in the manner

$$f(z,v,t)=f_o(v)\left[1+\epsilon(z)\sum_i^w a_i(t)H_i(\sqrt{av})\right], \quad (28)$$

where f_o is the Maxwell-Boltzmann law on velocities

$$f_o=n_o\left(\frac{a}{\pi}\right)^{1/2}\exp(-av^2); \quad a=\frac{m}{2k_B T_o}; \quad v=v_z, \quad (29)$$

with T_o the temperature of the medium, m the molecular mass, and K_B the Boltzmann constant. Here, n_o equals half number of particles per unit of volume for the equilibrium state. In principle, W is infinite and so is the number of coefficients a_i that needs to be determined. The Grad moment equations form an open hierarchy. Under suitable conditions, however, one can obtain reasonable results using a finite W value, as in the case we will discuss below, in which a suitable closure is introduced.

Assuming that (i) f is not too different from the equilibrium value f_o and (ii) the difference $f-f_o$ is produced by a plane wave of long wavelength ($k\ll 1$), we can write

$$f\approx f_o[1+b_o H_o+b_1 H_1+b_2 H_2], \quad (30)$$

where $b_j=b_j(z,t)=\epsilon(z)a_j(t)=\mu e^{ikz}a_j(t)$, μ being a small dimensionless parameter. Our goal here is that of determining b_o , b_1 , and b_2 . The first three Hermite polynomials are, respectively,

$$H_o=1, \quad H_1=2\sqrt{av}, \quad H_2=4av^2-2,$$

and Eq. (30) then takes the form

$$f=f_o[1+b_o+2b_1\sqrt{av}+2b_2(2av^2-1)]. \quad (31)$$

The coefficients b_j will be determined from appropriate velocity-moment conditions, for which we choose

$$\int_{-\infty}^{\infty} f(z,v,t)dv=n(z,t),$$

$$\int_{-\infty}^{\infty} (mv)f(z,v,t)dv=mn(z,t)u(z,t),$$

$$\int_{-\infty}^{\infty} \left(\frac{1}{2}m|v-u|^2\right)f(z,v,t)dv=\frac{1}{2}n(z,t)k_B T(z,t), \quad (32)$$

the magnitudes $n(z,t)$, $u(z,t)$, and $T(z,t)$ conserving their usual meaning, i.e., particle number per unit volume, bulk velocity, and temperature, respectively. The above equations are solvable subject to known initial conditions, for which we need to use the available *a priori* information.

Since the perturbation is small, we can use the LTE conservation theorems derived above for diluted monatomic gases in the preceding section. Thus, we assume that the (z,t) dependence of the MDF is given by the conservation theorems. Here, we are only concerned about the velocity dependence. Indeed, the whole purpose of the preceding section was to prepare the stage for the present MOM treatment.

To the first order in μ [see Eqs. (16) and (23)], we have (with $\rho=nm$)

$$n=n_o(1+\sigma)u=c_o\sigma T=T_o(1+\frac{2}{3}\sigma). \quad (33)$$

Thus, we have (to the same order of approximation)

$$nu \approx n_o c_o \sigma; \quad nT \approx n_o T_o (1 + \frac{5}{3} \sigma). \quad (34)$$

Inserting Eqs. (33) and (34) into the velocity-moment conditions, Eq. (32), we obtain

$$\int_{-\infty}^{\infty} f dv = n_o (1 + \sigma), \quad (35)$$

$$\int_{-\infty}^{\infty} v f dv = n_o c_o \sigma, \quad (36)$$

$$\int_{-\infty}^{\infty} v^2 f dv = \frac{n_o}{2a} \left(1 + \frac{5}{3} \sigma \right), \quad (37)$$

with a given by Eq. (29). Since f is given by Eq. (31), integration then yields an easily solvable linear set of equations for our unknowns b_o , b_1 , and b_2 :

$$n_o (1 + b_o) = n_o (1 + \sigma), \quad (38)$$

$$n_o \frac{1}{\sqrt{a}} b_1 = n_o c_o \sigma, \quad (39)$$

$$\frac{n_o}{2a} (1 + b_o + 4b_2) = \frac{n_o}{2a} \left(1 + \frac{5}{3} \sigma \right). \quad (40)$$

By solving the above system, one finds

$$b_o = \sigma; \quad b_1 = (\frac{5}{6})^{1/2} \sigma; \quad b_2 = \frac{1}{6} \sigma. \quad (41)$$

Finally, substituting Eqs. (41) into form (31) for f , and taking into account that $a = 5/(6c_o^2)$ [see Eqs. (20) and (29)], leads one to the molecular distribution function

$$f(z, v, t) = f_0(v) \left[1 + \sigma(z, t) \left(\frac{2}{3} + \frac{5}{3} \frac{v}{c_o} + \frac{5}{9} \frac{v^2}{c_o^2} \right) \right]. \quad (42)$$

The function $g(v)$ of Eq. (25) has now been determined to be

$$g(v) = \left(\frac{2}{3} + \frac{5}{3} \frac{v}{c_o} + \frac{5}{9} \frac{v^2}{c_o^2} \right), \quad (43)$$

which was our goal here.

According to Refs. [40,36,43], Eq. (42) is the BTE solution for our sound-propagation problem within the strictures posed by the order of approximation one is adopting here. We stress again the fact that the form of $\sigma(z, t)$ in Eq. (25) is derived from the above discussed LTE conservation theorems [cf. Eq. (22)]:

$$\sigma = \sigma_o \exp[i(kz - w_o t)], \quad c_o = \frac{w_o}{k}, \quad (44)$$

a sound wave of amplitude σ_o that propagates with velocity c_o .

V. SCHRÖDINGER WAVE EQUATION APPROACH TO NONEQUILIBRIUM THERMODYNAMICS

A. Connecting the Schrödinger wave equation's excited solutions to nonequilibrium thermodynamics: A brief review

The connection has been established in Ref. [27]. The excited solutions $\psi_n(x, t)$ to the Fisher-based SWE, Eq. (4), can be written as a superposition of Hermite-Gaussian polynomials of the form

$$\psi_n(x, t) = \phi_0(x) \sum_i b_{ni}(t) H_i(x), \quad n = 1, 2, \dots, \quad (45)$$

where ϕ_0 is the g.s. of the harmonic oscillator. The total number of coefficients $b_{ni}(t)$ depends upon how far from equilibrium we are. At equilibrium there is only one such coefficient. The coefficients $b_{ni}(t)$ are computed at the fixed time t at which the experimental input data $\langle A_k \rangle_t$ are collected. While the ground state solution of the Schrödinger wave equation gives the equilibrium states of thermodynamics [26], its excited solutions give nonequilibrium states [27]. In other words, $\psi_n(x, t)$ is connected, under certain conditions (see below), with the known solutions of the Boltzmann transport equation $f(x, t)$ by

$$f(x, t) = n(t) |\psi_n(x, t)|^2 \quad (46)$$

at a fixed time t , where $n(t)$ is the density of particles (whichever their velocity) at instant t .

The connection given by Eq. (46) was established in Ref. [27] via the Grad-MOM $f(x, t)$ of Eq. (27). Due to the fact that the square modulus of an expansion in Hermite-Gaussian polynomials is likewise a superposition of Hermite-Gaussian polynomials, with real coefficients $c_{in}(t)$,

$$|\psi_n(x, t)|^2 = [\phi_0(x)]^2 \sum_i c_{in}(t) H_i(x), \quad n = 1, 2, \dots, \quad (47)$$

then, for fixed n , the MOM coefficients $a_i(t)$ and $c_{in}(t)$ are connected. Indeed, both are certainly computed at a *fixed* time t . That is, the MOM momenta are evaluated at that time. Likewise, $\langle A_k \rangle$ of Eq. (2) can be regarded as velocity momenta at that time as well.

Although both the Grad-MOM and the Schrödinger wave equation coefficients are interconnected, an intrinsic difference exists between them, which is of a physical origin. The MOM moments at the time t are physically correct by construction, since one actually *solves* them using the Boltzmann transport equation. On the other hand, the premise of the constrained Fisher information approach is that its input constraints (here the velocity moments $\langle A_k \rangle_t$) are correct, since they are obtained from experiment. Here, we are assuming error-free values (the introduction of error bars will not be discussed here. We will deal with them in a future paper).

Summing up, the approach given in Ref. [27] gives solutions at the fixed (but arbitrary) time t which agree [via Eq. (46)] with those obtained using the MOM approach. This holds, for fixed n , at each time t [cf. Eq. (2)]. The Schrö-

dinger wave equation approach yields solutions valid at discrete time points t . In other words, for any other time value t^* we need to input new $\langle A_k \rangle$ values, appropriate for *this* time. The MOM, instead, obtains coefficients $a_i(t)$ valid for continuous time t , since they are using the Boltzmann transport equation, which is a continuous one. This distinction, “discrete versus continuous,” does not compromise the validity of the Fisher-Schrödinger \Leftrightarrow nonequilibrium thermodynamics bridge.

B. The Schrödinger wave equation treatment

In this section, we will analyze the propagation of sound waves in a dilute gas using the Fisher information technique. The solution so obtained will be compared with that of the Boltzmann transport equation.

In order to find the PDF one has to solve the Schrödinger wave equation corresponding to our available prior information. Due to the fact that the sound wave is traveling along the z direction, we choose the x coordinates so that Eq. (4) becomes $p_z = \rho_o v$ (v denoting v_z) and write the Schrödinger wave equation in the velocity space as

$$-\frac{1}{2\rho_o} \frac{\partial^2 \psi}{\partial v^2} + \sum_{k=1}^M \lambda'_k(t) A_k(v) \psi = \frac{\alpha}{8} \psi, \quad (48)$$

where $\lambda'_k(t) A_k(v) = -(1/8) \lambda_k(t) A_k(p_z)$. The procedure to be followed involves the following two stages.

(1) We determine first the global-equilibrium state that is to be later perturbed by the sound wave. Our input empirical information at this stage is that of n_o , T_o , and ρ_o . Via the equipartition theorem this is tantamount to knowing $\langle v^2 \rangle$, the one and only expectation value that enters here Eq. (48).

(2) In the second stage, we use experimental facts regarding sound propagation in a fluid [empirical hydrodynamic equations (11)–(13)]. This tells us that the (z, t) dependence of the density perturbation produced by the sound wave is of the form prescribed in Eqs. (15) and (22). Additionally, we assume knowledge of the empirical velocity of sound in the medium. As discussed below, this entails knowledge of $\langle v \rangle$.

1. Ground state

At first stage, we deal with a monatomic gas that it is globally at rest. We know the gas density ρ_o , its temperature T_o , and the number of molecules per unit volume n_o . Using the equipartition theorem, we can write

$$\langle \frac{1}{2} \rho_o v^2 \rangle_o = \frac{1}{2} n_o k_B T_o, \quad (49)$$

where the notation $\langle \rangle_o$ is utilized to indicate that the average value is calculated in the equilibrium state. Equation (49) constitutes the prior knowledge. Therefore, choosing $A_1(v) = v^2$ and writing $\lambda'_1(t) = \rho_o / (2\omega_o^2)$, $\alpha/8 = E/\omega_o^2$, the time-independent Schrödinger wave equation is given by

$$\left[-\frac{1}{2\rho_o} \frac{\partial^2}{\partial v^2} + \frac{\rho_o}{2\omega_o^2} v^2 \right] \psi = \frac{E}{\omega_o^2} \psi \quad (50)$$

which can be recast as

$$\frac{\omega_o}{2} \left[-\frac{1}{a} \frac{\partial^2}{\partial v^2} + av^2 \right] \psi = E \psi, \quad (51)$$

where $a = \rho_o / \omega_o$. Equation (51) is the well-known Schrödinger equation for the harmonic oscillator, for which the ground state solution is given by

$$\phi_o = \left(\frac{a}{\pi} \right)^{1/4} \exp\left(-\frac{a}{2} v^2 \right). \quad (52)$$

The prior knowledge, Eq. (49), then leads to

$$a = \frac{\rho_o}{\omega_o} = \frac{m}{2k_B T_o}. \quad (53)$$

Obviously, the ground state solution of the Schrödinger wave equation is connected with the thermal equilibrium state

$$n_o \phi_o^2 = f_o, \quad (54)$$

where f_o is the first term of the MOM expansion, i.e., the Maxwell Boltzmann law.

2. Superposition of ground plus excited states

(a) *Setting up the appropriate SWE problem.* One is concerned here with a sound wave of long wavelength ($k \ll 1$) that traverses a monatomic gas. According to our scenario, we are in possession of two important pieces of information, namely, that the sound velocity c_o is independent of the frequency [41] and that the accompanying internal processes are adiabatic [41], i.e.,

$$c_o^2 = \gamma \frac{k_B T_o}{m}, \quad \gamma = 5/3. \quad (55)$$

We now assume that a further additional piece of knowledge is that of the actual (experimental) value for the gas sound velocity c_o . As we have seen in Sec. III A that $u \propto c_o$, this entails knowing the mean value of v . According to our CFIE point of view, we should now introduce a linear term in v within the “information” potential entering the Fisher SWE, accompanied by an appropriate Lagrange multiplier χ which, for the sake of convenience, is given the form $\chi = \sqrt{a} \vartheta$. We then need to determine ϑ .

Our Fisher-Schrödinger wave equation now becomes

$$\frac{\omega_o}{2} \left[-\frac{1}{a} \frac{\partial^2}{\partial v^2} + av^2 + \vartheta \sqrt{a} v \right] \psi = E \psi \quad (56)$$

that can be treated perturbatively in view of our knowledge of the problem.

It is well known that if one perturbs the ground state of the harmonic oscillator with a linear term, only the first excited state enters the perturbative series because of the pertinent selection rules [45], i.e.,

$$\langle \psi_0 H_n(z) | z | \psi_0 H_m(z) \rangle = c_1 \delta(n, m+1) + c_2 \delta(n, m-1), \quad (57)$$

where c_1 and c_2 are appropriate constants, which entails that for $m=0$ (ground state), only $n=1$ (first excited state) contributes.

We can thus write Eq. (45) in the manner

$$\psi \approx \phi_0 [d_0 H_0 + d_1 H_1], \quad (58)$$

where we write $d_j = d_j(z, t) = \mu e^{ikz} b_j(t)$ ($\mu \ll 1$). The first two Hermite polynomials are

$$H_0 = 1, \quad H_1 = 2\sqrt{av}.$$

We can thus cast Eq. (58) in the form

$$\psi = \phi_0 [d_0 + 2d_1 \sqrt{av}]. \quad (59)$$

Thus we have

$$|\psi|^2 = \phi_0^2 [\alpha_0 + \alpha_1 2\sqrt{av} + \alpha_2 4av^2], \quad (60)$$

where the real coefficients α_i are related to the d_i by the following expressions:

$$\alpha_0 = |d_0|^2; \quad \alpha_1 = 2(d_0^* d_1 + d_0 d_1^*); \quad \alpha_2 = |d_1|^2. \quad (61)$$

The coefficients α_j will be determined from the appropriate velocity-momenta conditions. We start by calculating

$$\langle v^n \rangle = \int_{-\infty}^{\infty} v^n |\psi|^2 dv, \quad n = 0, 1, 2 \quad (62)$$

using Eq. (60). We obtain

$$\begin{aligned} \langle 1 \rangle &= \alpha_0 + 2\alpha_2, \\ \langle v \rangle &= \frac{1}{\sqrt{a}} \alpha_1, \\ \langle v^2 \rangle &= \frac{1}{2a} (\alpha_0 + 6\alpha_2). \end{aligned} \quad (63)$$

On the other hand, we can use the *a priori* information for evaluating the above mean values.

(b) *Remarks on the treatment of information in the two different approaches.* At this stage, it is important to point out that we are looking at things in a way that is quite different from that of Sec. III. Here, the system of equations formed by (i) the continuity equation, Eq. (11), (ii) the Euler's equation (12), and (iii) the experimental gas state equation $P = P(\rho)$ [cf. Eqs. (17) or (14)] are regarded as empirical hydrodynamic equations that describe the propagation of sound in a nonviscous gas [44]. This notion was already advanced in the paragraph following Eq. (12). Notice that our Fisher treatment is a proper thermodynamic one [26,27] and as such does not require any input information that would assume underlying microscopic structures. This would indeed have been the case if we tried to include any information from microscopic models such as the LTE or even the

BTE for colliding atoms. However, as the input experimental information indicates that we are in the presence of an adiabatic process [and thus the equation of state (17) applies], our LTE conservation theorems and the empirical hydrodynamic equations necessarily do agree.

The MOM approach uses the arguments and derivations of Sec. III A so as to obtain the velocity moments. There the LTE conservation theorems constituted the source from which the relevant information derived. In the Fisher instance, the same information is used once again, but with a twist: it is regarded as derived from experiment, not from conservation theorems. In both instances we obtain, as a result of judiciously employing the *a priori* information, the (z, t) dependence of the density perturbation σ in $\rho = \rho_0(1 + \sigma)$ [cf. Eq. (15)]. What we need to determine is the velocity dependence of the distribution function.

(c) *Solving the Fisher-SWE problem.* Now, up to the first order in μ , we have [see Eqs. (16) and (23)], using $\rho = nm$,

$$n = n_0(1 + \sigma); \quad u = c_0 \sigma; \quad T = T_0(1 + \frac{2}{3}\sigma). \quad (64)$$

Moreover, at the same order of approximation in μ , we get

$$\langle v^2 \rangle = \frac{k_B T}{m} \sim \frac{1}{2a} \left(1 + \frac{2}{3}\sigma \right), \quad (65)$$

where we have invoked both the equipartition theorem and the knowledge that the process is adiabatic.

Now taking into account, with reference to Eqs. (63), both the normalization condition and Eqs. (64) and (65), we get

$$\begin{aligned} 1 &= \alpha_0 + 2\alpha_2, \\ c_0 \sigma &= \frac{1}{\sqrt{a}} \alpha_1, \\ 1 + \frac{2}{3}\sigma &= \alpha_0 + 6\alpha_2. \end{aligned} \quad (66)$$

The above equations' system leads to

$$\alpha_0 = 1 - \frac{1}{3}\sigma, \quad \alpha_1 = c_0 \sqrt{a} \sigma, \quad \alpha_2 = \frac{1}{6}\sigma. \quad (67)$$

Substituting then Eq. (67) into Eq. (60), we find

$$|\psi|^2 = \phi_0^2 \left[1 + \sigma \left(-\frac{1}{3} + \frac{5}{3} \frac{v}{c_0} + \frac{5}{9} \frac{v^2}{c_0^2} \right) \right]. \quad (68)$$

Inserting now Eq. (67) into Eq. (61) leads to

$$\begin{aligned} d_0 &= (1 - \frac{1}{6}\sigma) e^{i\alpha_0}, \quad d_1 = \sqrt{\frac{1}{6}\sigma} e^{i\alpha_1}, \\ \cos(\alpha_0 - \alpha_1) &= \frac{1}{2} \sqrt{5\sigma}, \end{aligned} \quad (69)$$

so that, substituting the above expression into Eq. (59), we obtain

$$\psi = \phi_o \left[1 - \frac{\sigma}{6} \left(1 - 5 \frac{v}{c_o} \right) - i \frac{1}{3} \sqrt{5} \sigma \frac{v}{c_o} \right], \quad (70)$$

where quantities smaller than those of the first order in μ have been neglected. Finally, we obtain the value $\vartheta = -(2/3)\sqrt{6/5}$ for the Lagrange multiplier associated to our knowing the actual value of the sound's velocity [cf. Eq. (56)].

At this stage, the reader may well question the interest in the wave function (70), insofar as there seems to be nothing relevant in ψ that is not already contained in the probability distribution function f . Three points are worth emphasizing in this respect, given as follows.

(1) Wooters has shown [46] that the *distinguishability* between two probability distributions f_1 and f_2 is measured by the *angle* in the Hilbert space between the corresponding amplitudes (square roots) ψ_1 and ψ_2 . More specifically, let us define Ω as the appropriate space of probability distributions we are dealing with. Then, for $f_1, f_2 \in \Omega$ we define the Wootters distance as

$$\mathcal{D}_W(f_1, f_2) = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} Y_n, \quad (71)$$

where Y_n is the maximum number of intermediate vectors f mutually distinguishable in n trials [46]. From our informational viewpoint, we are always going to infer PDF's out of input data that will be contaminated by error. Physical distinguishability of PDF's then constitutes an important issue.

(2) No equation has received more attention with a view to approximations and numerical techniques than the SWE. Thus, if faced with a given equation, it is always worthwhile to transform it, if at all possible, into a SWE. As an example, see the treatment by Prigogine of the Liouville equation in Ref. [47].

(3) In addition to these two purely practical points, there is the important theoretical result described in the Introduction that both thermodynamics and quantum mechanics can be expressed by a recourse to a formal SWE, out of a common informational basis [14].

C. The connection

We are now in a position to appreciate just how the excited states' solution to the Schrödinger wave equation is connected with our thermal off-equilibrium, sound-propagation state. The link reads, setting $f(x, t) = n(x, t) |\psi|^2$,

$$f = n_o (1 + \sigma) \phi_o^2 \left[1 + \sigma \left(-\frac{1}{3} + \frac{5}{3} \frac{v}{c_o} + \frac{5}{9} \frac{v^2}{c_o^2} \right) \right], \quad (72)$$

that is,

$$f(x, t) = n_o \phi_o^2 \left[1 + \sigma \left(\frac{2}{3} + \frac{5}{3} \frac{v}{c_o} + \frac{5}{9} \frac{v^2}{c_o^2} \right) \right]. \quad (73)$$

Due to the fact that $n_o \phi_o^2 = f_0$ [see Eq. (54)], it is clear that

$$n |\psi|^2 = f, \quad (74)$$

where f is the molecular distribution function obtained via the MOM technique [see Eq. (42)]. This is an elegant and illuminating relationship (a kind of Rosetta stone) connecting the off-equilibrium sound-propagation state with a superposition of ground plus excited states of a Schrödinger wave equation. The molecular distribution for the off-equilibrium sound-propagation state is proportional to the square modulus of a Hilbert space wave function. Of course, after making this connection we are now able to speak of a microscopic substructure also within the Fisher context.

VI. CONCLUSIONS

We have revisited in this work the Fisher-SWE technique developed in Ref. [27]. We state now our main conclusions below.

(1) As a rather interesting result, we have here illustrated, with reference to the sound wave propagation process in dilute gases and its treatment via the BTE, the equivalence between Grad's method [36,40,43] for the BTE and the Fisher SWE approach [27] that does not use (at all) the BTE.

(2) Additionally, we show how to derive, using these two methods, the (z, v, t) dependence of the perturbation to the fluid density originated by a sound wave propagating along the z axis. Thus, starting with appropriate information, two different methods are given in order to obtain the pertinent MDF that represents the sound-propagation state of the dilute gas. These are given below.

(A) The Boltzmann method: The LTE conceptual information and BTE conservation theorems constitute the Grad BTE approach, which leads to the MDF.

(B) The Fisher method: The hydrodynamic experimental information (in two stages) constitutes the Fisher-Schrödinger wave equation, which again leads to the MDF. Note that this method is independent of the Grad method.

It is important to reiterate that the *source* of prior information differs for the two approaches. In method (A) we have included, as an essential form of prior information, the LTE conservation theorems. In method (B) instead, we had to do with hydrodynamic experimental information, in two stages. In the first stage, we assume experimental knowledge of density, temperature, and molecular weight that enables one to use the equipartition theorem. In the second stage, we assume having measured the actual velocity of sound.

(3) Remember that thermodynamics can be regarded, from an axiomatic viewpoint, as a logical mathematical structure whose axioms are empirical results [37]. Thus, thermodynamics does not, indeed cannot, assume any underlying microscopic picture. Since our Fisher treatment is a proper thermodynamic one [26,27], it does not require any input information that would assume underlying microscopic structures, which would indeed have been the case if we tried to include any information from microscopic models such as the LTE. However, as the input experimental information indicates that we are in the presence of an adiabatic process [and thus the equation of state (17) applies], the LTE conservation theorems and the empirical hydrodynamic equations

necessarily do agree, and, although the origin of the input information is different in method (A) and (B), the output informational content turns out to be equivalent in both approaches.

The *a priori* information in both approaches (A) and (B) determines the (z,t) dependence of the molecular distribution function. The velocity dependence $g(v)$ is obtained in quite disparate ways in methods (A) and (B). They do coincide in the final result, though. As far as new physics is concerned, this fact constitutes the main contribution of the present paper.

(4) From an epistemological point of view, one should contrast the transparency and the immediacy of the Fisher approach with the convoluted and arduous path one must traverse in the BTE case, as the Appendix dramatically and unequivocally illustrates.

(5) After connecting the MOM with the Fisher results we are entitled to discuss the underlying substructures within the latter approach. Even though, according to the constrained Fisher tenets, we expected that the MDF would be given by the square modulus of some SWE wave function (WF), what we actually found is that this WF, for sound propagation in a dilute gas, is identical to that of a harmonic oscillator perturbed by a linear term, a rather surprising result, since just *one* excited state enters the concomitant linear superposition. Situations that require higher-order perturbative treatments are currently under investigation and are planned to be published elsewhere.

APPENDIX: CONSERVATION THEOREMS FOR THE BOLTZMANN EQUATION

We remind the reader in this appendix of the main factors entering the discussion of the conservation laws involved in a discussion of (exclusively) the Boltzmann transport equation [41]. None of the following considerations apply to the Fisher treatment. The BTE (without external forces) reads

$$\left[\frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{r}} \right] f = D_c f, \quad (\text{A1})$$

where $f = f(\mathbf{r}, \mathbf{v}, t)$ is the MDF and $D_c f$ represents changes in f due to collision effects, whose explicit expression can be found in Ref. [40]). As we saw in Sec. III, in any molecular collision there are dynamical quantities that are rigorously conserved, which gives rise to important conservation theorems.

If (i) $\xi(\mathbf{r}, \mathbf{v})$ is any quantity associated with a molecule of velocity \mathbf{v} located at \mathbf{r} , and, additionally (ii) in any collision $\mathbf{v}_1, \mathbf{v}_2 \rightarrow \mathbf{v}'_1, \mathbf{v}'_2$, taking place at \mathbf{r} , we have $\xi_1 + \xi_2 = \xi'_1 + \xi'_2$, then (iii) ξ is a conserved quantity and [41]

$$\int d^3v \xi(\mathbf{r}, \mathbf{v}) [D_c f] = 0. \quad (\text{A2})$$

From Eq. (A2) and an appropriate choice of ξ , we get three independent conservation theorems for mass, momentum, and energy, respectively, [41].

1. The relevant quantities

If we know the BTE solution $f = f(\mathbf{r}, \mathbf{v}, t)$, we can immediately evaluate the following quantities:

$$\rho(\mathbf{r}, t) = m \int d^3v f(\mathbf{r}, \mathbf{v}, t) \quad (\text{mass density}), \quad (\text{A3})$$

$$\mathbf{u}(\mathbf{r}, t) = \int d^3v f(\mathbf{r}, \mathbf{v}, t) \mathbf{v} \quad (\text{bulk velocity}), \quad (\text{A4})$$

$$\Theta(\mathbf{r}, t) = \frac{m}{3} \langle |\mathbf{v} - \mathbf{u}|^2 \rangle \quad (k_B \text{ times temperature}), \quad (\text{A5})$$

$$\mathbf{q}(\mathbf{r}, t) = \frac{m}{3} \rho \langle |\mathbf{v} - \mathbf{u}|^2 (\mathbf{v} - \mathbf{u}) \rangle \quad (\text{heat flux vector}), \quad (\text{A6})$$

$$P_{ij} = \rho \langle [(v_i - u_i)(v_j - u_j)] \rangle \quad (\text{pressure tensor}), \quad (\text{A7})$$

$$n(\mathbf{r}, t) = \int d^3v f(\mathbf{r}, \mathbf{v}, t) \quad (\text{particle density}). \quad (\text{A8})$$

2. Conservation theorem

The BTE implies [41]

$$\frac{\partial}{\partial t} \langle n \xi \rangle + \frac{\partial}{\partial x_i} \langle n v_i \xi \rangle - n \left\langle v_i \frac{\partial \xi}{\partial x_i} \right\rangle = 0, \quad (\text{A9})$$

where Einstein's subindex convention is employed. Let us now choose ξ to be $\xi = m$ (mass), $\xi = m v_i$ (momentum), and $\xi = (m/2) \langle |\mathbf{v} - \mathbf{u}|^2 \rangle$ (thermal energy). Using Eq. (A9) one immediately obtains three conservation laws, namely, those of (i) mass, (ii) momentum, and (iii) energy [41].

Of course, as remarked by Huang [41], in the classical treatment (not in the Fisher one) one needs first to solve the BTE and find then the pertinent MDF to be in a position of verifying that the above conservation laws are indeed fulfilled. This does not seem to be of much help. In text, we have used the local thermal equilibrium framework to construct $f^{(0)}$. Inserting it into Eq. (A9), the above conservation laws yield now three conditions that \mathbf{u} , ρ , and T must verify, namely, Eqs. (11)–(13). They are assumed in the classical textbook treatment [41] to vary in a slow, smooth manner according to the LTE. We learn now that this variation must respect these conservation equations. Moreover, our three guiding Eqs. (11)–(13) turn out to be the hydrodynamic equations for the nonviscous flow of a gas [41]. Two additional results are derived from these considerations [41]: (i) $\mathbf{q}(\mathbf{r}, t)$ vanishes identically, so that all our local processes are adiabatic, and (ii) $P = n k_B T$.

All the material in this appendix pertains exclusively to the Boltzmann treatment, and none of it whatsoever is needed in the Fisher treatment.

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